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The singular locus of a Schubert variety

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Abstract

The singular locus of a Schubert variety X_μ in the flag variety for $GL_n(\mathbf{C})$ is the union of Schubert varieties X_ν, where ν runs over a set $\mathrm{Sg}(\mu)$ of permutations in S_n. We describe completely the maximal elements of $\mathrm{Sg}(\mu)$ under the Bruhat order, thus determining the irreducible components of the singular locus of X_μ.

Résumé

Le lieu singulier d'une variété de Schubert X_μ de la variété des drapeaux de $GL_n(\mathbf{C})$ est réunion de variétés de Schubert X_ν où ν parcourt un ensemble $\mathrm{Sg}(\mu)$ de permutations de S_n. Nous décrivons les éléments maximaux de $\mathrm{Sg}(\mu)$ pour l'ordre de Bruhat, ce qui détermine complètement les composantes irréductibles du lieu singulier de X_μ.

Keywords: Schubert variety; Singular locus; Permutations; Diagrams

Mots-clés : Variété de Schubert ; Singularité ; Lieu singulier ; Permutations ; Diagrammes

Introduction

The aim of this article is to describe completely the singular loci of the Schubert varieties of type A, i.e., those appearing in the flag variety of $GL_n(\mathbf{C})$. As is well known,

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such Schubert varieties are indexed by the symmetric group S_n. Schubert varieties are not smooth in general and it has been a long-standing open problem to determine the irreducible components of their singular loci (cf. [BL] for a detailed history of this problem and a list of partial solutions).

The singular locus of the Schubert variety X_μ corresponding to an element $\mu \in S_n$ is the union of smaller Schubert varieties X_ν, where ν runs over a set $\mathrm{Sg}(\mu)$ of permutations in S_n. The maximal elements of $\mathrm{Sg}(\mu)$ under the Bruhat order correspond to the irreducible components of X_μ. With Theorem 1.3, which is our main result, we determine the singular locus of X_μ by characterizing the maximal elements of $\mathrm{Sg}(\mu)$. Our characterization is in terms of certain planar representations. We show that the maximal elements of $\mathrm{Sg}(\mu)$ fall into three types, each of them being invariant under taking inverses and under conjugation by the longest element of S_n.

Lakshmibai and Sandhya had defined a subset Z_μ of S_n and conjectured in [LSa] that the set of maximal elements of Z_μ coincides with the set of maximal elements of $\mathrm{Sg}(\mu)$. Gasharov [Ga] showed that Lakshmibai and Sandhya's singularity conditions were sufficient. Using Theorem 1.3, we establish that they are necessary, thus proving Lakshmibai and Sandhya's conjecture.

In the proof of Theorem 1.3 we use two main technical tools. The first one is what we call the diagram of a couple (ν, μ) of permutations in S_n. This consists of a certain square tableau of $(n-1)^2$ integers together with "circuits" linking the $2n$ points of the graphs of ν and μ (see Section 2.1 for a precise definition). The integers in the diagram of (ν, μ) are all non-negative if and only if $\nu \leqslant \mu$ under the Bruhat order.

Flippable versions are our second tool. By analogy with the inversions of a permutation, we call version of ν any pair (i, j) of integers such that $i < j$ and $\nu(i) < \nu(j)$. A version (i, j) of ν is flippable with respect to a permutation $\mu > \nu$ if the composition of ν with the transposition of i and j is still $\leqslant \mu$. Flippable versions can easily be detected on the diagram of (ν, μ). The dimension of the tangent space of the Schubert variety X_μ at a generic point of $X_\nu \subset X_\mu$ can be computed from the number of versions of ν that are not flippable with respect to μ. It follows that a permutation ν belongs to $\mathrm{Sg}(\mu)$ if and only if the number of non-flippable versions of ν is less than the number of versions of μ.

The paper is divided into eleven sections. In Section 1 we define three types $\mathrm{I}(a, b)$, $\mathrm{I}(n)$, $\mathrm{II}(a, b)$ of permutations and give the main theorem (Theorem 1.3), which states that the maximal elements of $\mathrm{Sg}(\mu)$ are of these types. We define the diagram of a couple of permutations in Section 2 and state its main properties. In Section 3 we prove Lakshmibai and Sandhya's conjecture. Flippable versions are defined in Section 4 and we characterize maximal elements of $\mathrm{Sg}(\mu)$ in terms of non-flippable versions; this immediately allows us to prove that any permutation ν belonging to one of the types $\mathrm{I}(a, b)$, $\mathrm{I}(n)$, or $\mathrm{II}(a, b)$ is a maximal element of $\mathrm{Sg}(\mu)$.

The rest of the paper is devoted to proving the converse, which is the more difficult part of Theorem 1.3. In Section 5 we characterize jump permutations, namely those where the dimension of the tangent space of X_μ drops after flipping some flippable version. We use this in Section 6 to show that the diagram of (ν, μ) where ν is a maximal element of $\mathrm{Sg}(\mu)$ has what we call a 1324-configuration or a 2143-configuration; this implies in particular that such a permutation ν has a 1324 or a 2143 pattern. In Sections 7 and 8 we define maximal positive rectangles in diagrams and we show that a number of situations cannot occur if ν is a maximal element of $\mathrm{Sg}(\mu)$. In Section 9 we prove that the graph of a maximal element of $\mathrm{Sg}(\mu)$ has the shape required by Theorem 1.3. In Section 10 we show that any maximal element of $\mathrm{Sg}(\mu)$ with a 1324-configuration is of type $\mathrm{I}(a,b)$ or $\mathrm{I}(n)$. We complete the proof of Theorem 1.3 in Section 11 by proving that any maximal element of $\mathrm{Sg}(\mu)$ with a 2143-configuration, but no 1324-configuration, is of type $\mathrm{II}(a,b)$.

We thank V. Lakshmibai for providing us with Gasharov's preprint [Ga]. After this work was completed, we learned that Billey–Warrington [BW] and Manivel [Ma1] have also recently solved the problem of determining the singular locus of a Schubert variety.

1. The main result

1.1. Schubert varieties

Let $G = GL_n(\mathbf{C})$ be the group of complex invertible $(n \times n)$-matrices and B the subgroup of upper triangular matrices in G. The Bruhat decomposition of G with respect to B induces a partition of the flag variety G/B into Schubert cells C_μ indexed by the elements μ of the symmetric group S_n on $\{1, 2, \dots, n\}$:

$$G/B = \coprod_{\mu \in S_n} C_\mu.$$

The flag variety has a natural structure of a complex projective algebraic variety of dimension $n(n-1)/2$. By definition, the *Schubert variety* X_μ associated to $\mu \in S_n$ is the Zariski closure of C_μ in G/B. The dimension of X_μ is equal to the length $\lg(\mu)$ of μ with respect to the set of simple transpositions $\tau_{i,i+1}$ in S_n. Each Schubert variety X_μ is a disjoint union of Schubert cells, namely, of those cells C_ν for which $\nu \leqslant \mu$ for the Bruhat order $\leqslant$ on S_n:

$$X_\mu = \coprod_{\nu \leqslant \mu \in S_n} C_\nu.$$

It follows that $X_\nu \subset X_\mu$ if and only if $\nu \leqslant \mu$.

As complex algebraic varieties, Schubert varieties are not smooth in general. Lakshmibai and Sandhya [LSa] showed that X_μ is smooth if and only if the permutation $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ avoids the patterns 3412 and 4231, i.e., there are no integers $1 \leqslant i < j < k < \ell \leqslant n$ such that

$$\mu_k < \mu_\ell < \mu_i < \mu_j \quad \text{or} \quad \mu_\ell < \mu_j < \mu_k < \mu_i.$$

The singular locus $\mathrm{Sing}\, X_\mu$ of a Schubert variety is a union of Schubert subvarieties. Let $\mathrm{Sg}(\mu)$ be the subset of elements $\nu \in S_n$ such that X_ν is contained in $\mathrm{Sing}\, X_\mu$. We say that $\nu \in S_n$ is *singular with respect to* μ or that the *couple* (ν, μ) *is singular* if ν belongs to $\mathrm{Sg}(\mu)$.

Let $\mathrm{Msg}(\mu)$ be the subset of maximal elements of $\mathrm{Sg}(\mu)$ for the Bruhat order. An element of $\mathrm{Msg}(\mu)$ will be said to be *maximal singular with respect to* μ; we will also say that the *couple* (ν, μ) *is maximal singular.* Since $\nu' < \nu$ and $\nu \in \mathrm{Sg}(\mu)$ implies $\nu' \in \mathrm{Sg}(\mu)$, we see that $\mathrm{Msg}(\mu)$ completely determines the singular locus of X_μ:

$$\operatorname{Sing} X_\mu = \bigcup_{\nu \in \mathrm{Msg}(\mu)} X_\nu.$$

The problem we solve in this paper is to determine the set $\mathrm{Msg}(\mu)$ for any $\mu \in S_n$.

1.2. Planar representations and circuits

To permutations in S_n we will associate certain planar configurations, all placed in the square $[1, n]^2$ in $\mathbf{R}^2$. As with matrices, when we represent graphically such a square, the rows increase from top to bottom and the columns from left to right. When we refer to symmetries in the sequel, we mean the orthogonal symmetries with respect to the diagonals and to the center of the square.

To any permutation $\mu \in S_n$ we associate its *graph*, which is the subset of all points $(i, \mu(i))$ in $\{1, 2, \ldots, n\}^2$, where $i = 1, \ldots, n$. The set $\{1, 2, \ldots, n\}^2$ will be given its natural partial order induced from the natural order on $\{1, 2, \ldots, n\}$. Therefore, we shall be able to speak about comparable points of the graph of μ.

To a couple (ν, μ) of permutations in S_n we associate what we call the planar representation of (ν, μ). The graphs of both ν and μ are part of the planar representation; to distinguish one graph from the other in the figures, we represent each point of the graph of ν (respectively of μ) by a cross $\times$ (respectively by a circle $\circ$). When $\mu(i) = \nu(i)$ for some i, we have what we call a *double point,* namely a circle and a cross in the same spot, in which case we draw the symbol $\otimes$. A point that is not double will be called *simple.*

If $\mu(i) \neq \nu(i)$, we define the non-trivial horizontal segment $H_i = \{i\} \times [\mu(i), \nu(i)]$ and the non-trivial vertical segment $V_i = [i, \mu^{-1}\nu(i)] \times \{\nu(i)\}$. We consider the union in $[1, n]^2$ of the segments H_i and V_i, where i runs over all integers such that $\mu(i) \neq \nu(i)$, and call it the *boundary* of the planar representation. By definition, the *planar representation* of the couple (ν, μ) is the configuration consisting of the graphs of ν and μ (marked by $\times$ and $\circ$ as indicated above) and of the boundary (represented in all figures by thick lines). Observe that the boundary of the planar representation is the union of closed polygonal arcs whose vertices are crosses and circles alternatingly. Such a closed polygonal arc is called a *circuit* for (ν, μ). It is easy to see that the number of circuits for (ν, μ) is equal to the number of non-trivial cycles of the permutation $\mu^{-1}\nu$. (Note that the point $(i, \mu(i))$ belongs to the same circuit as the point $(\mu^{-1}\nu(i), \mu(\mu^{-1}\nu(i)))$.) The trivial cycles of $\mu^{-1}\nu$, i.e., its fixed points, are in bijection with the double points of the planar representation.

See Figs. 1.1–1.3 and 2.1 for examples of planar representations. The planar representation in Fig. 2.1 has exactly two circuits (one of them is a simple curve, i.e., with no self-intersection, the other one has one self-intersection).

We now consider three important types of planar representations with a unique circuit, which is a simple curve:

(i) Type $\mathrm{I}(a, b)$, where a and b are integers $\geqslant 1$. A typical planar representation of type $\mathrm{I}(a, b)$ is represented in Fig. 1.1; the integer a (respectively b) is the number

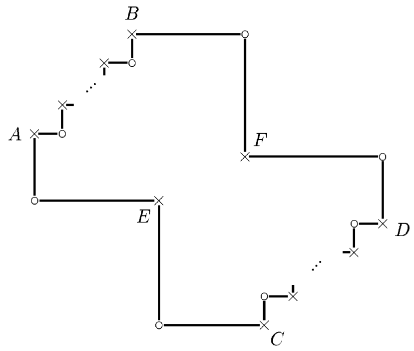


Fig. 1.1.

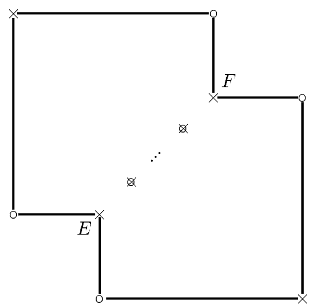


Fig. 1.2.

of crosses in the upper left corner (respectively the lower right corner) of the figure. If $a = 1$, then the points A and B in the figure coincide and there are no crosses between them. Similarly, if $b = 1$, then the points C and D coincide and there are no crosses between them. The boundary of the planar representation consists of a unique circuit. The planar representation may have double points outside the surface bounded by the circuit (these double points are not drawn on the figure), but no double point may lie inside the circuit. Let $(x_A, y_A), \ldots, (x_F, y_F)$ be the respective coordinates of the points $A, \ldots, F$. Then we must have

$$x_B \leqslant x_A < x_F < x_E < x_D \leqslant x_C \quad \text{and} \quad y_A \leqslant y_B < y_E < y_F < y_C \leqslant y_D.$$

(ii) Type $\mathrm{I}(n)$, where n is an integer $\geqslant 0$. A typical planar representation of type $\mathrm{I}(n)$ is represented in Fig. 1.2; the integer n is the number of double points inside the surface bounded by the one-circuit boundary of the planar representation (if $n = 0$, there are none). These double points are placed in such a way that none is comparable to another one, to E or to F for the partial order on $\{1, \ldots, n\}^2$. Other double points (not drawn) may lie outside the circuit.

(iii) Type $\mathrm{II}(a, b)$, where a and b are integers $\geqslant 2$. A typical planar representation of type $\mathrm{II}(a, b)$ is represented in Fig. 1.3; the integer a (respectively b) is the number of crosses in the upper left corner (respectively the lower right corner) of the figure.

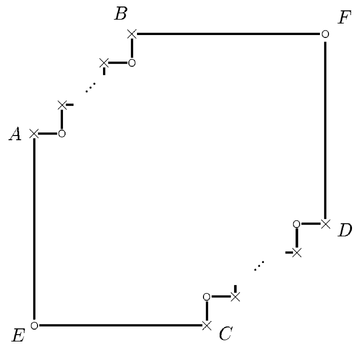


Fig. 1.3.

The boundary of the planar representation consists of a unique circuit. The planar representation may have double points outside the surface bounded by the circuit (these double points are not drawn on the figure), but no double point may lie inside the circuit. Let $(x_A, y_A), \dots, (x_F, y_F)$ be the respective coordinates of the points $A, \dots, F$. Then we must have

$$x_F = x_B < x_A < x_D < x_C = x_E \quad \text{and} \quad y_E = y_A < y_B < y_C < y_D = y_F.$$

Observe that types $\mathrm{I}(a, b)$, $\mathrm{I}(n)$, or $\mathrm{II}(a, b)$ are symmetric with respect to the diagonals and the center of the square $[1, n]^2$.

We now state the main theorem of the article.

1.3. Theorem. *Given $\mu \in S_n$, a permutation ν is maximal singular with respect to μ if and only if the planar representation of the couple (ν, μ) is of type $\mathrm{I}(a, b)$, $\mathrm{I}(n)$, or $\mathrm{II}(a, b)$ for some integers a, b, n.*

1.4. Corollary. *If (ν, μ) is maximal singular, then the permutation $\nu\mu^{-1}$ has a unique non-trivial cycle.*

1.5. Corollary. *If one removes double points from the representation of a maximal singular couple (ν, μ), then the couple (ν', μ') of restricted permutations is also maximal singular.*

1.6. Remark. The implications converse to Corollaries 1.4 and 1.5 do not hold, as can be deduced from $\nu = (2, 1, 3, 5, 4)$ and $\mu = (5, 2, 3, 4, 1)$.

1.7. Remark. In case of a permutation μ avoiding the pattern 3412, one can compute all Kazhdan–Lusztig polynomials $P_{\mu,\nu}$ explicitly (cf. [Lc]). This description provides in particular the maximal singular permutations with respect to such a μ; they are all of type $\mathrm{II}(a, b)$. Cortez [Col] has given an explicit geometrical description of generic singularities in this case (and, simultaneously with Manivel [Ma2], extended this description to the general case in [Co2]).

2. Diagram of a couple of permutations

2.1. North–east diagrams

Given a couple (ν, μ) of permutations in S_n, we define its *NE-diagram* (also called its *diagram*) as the tableau consisting of the planar representation of (ν, μ) together with $(n-1)^2$ integers placed each in each square delimited by the union $\{1, \dots, n\} \times [1, n] \cup [1, n] \times \{1, \dots, n\}$ of horizontal and vertical segments in $[1, n]^2$. For $1 \leqslant p < n$ and $1 \leqslant q < n$ the integer $\mathrm{NE}_{\nu,\mu}(p, q)$ assigned to the square $[p, p+1] \times [q, q+1]$ is given by

$$\mathrm{NE}_{\nu,\mu}(p, q) = \mathrm{card}\bigl\{k \leqslant p \bigm| \mu(k) > q\bigr\} - \mathrm{card}\bigl\{k \leqslant p \bigm| \nu(k) > q\bigr\}.$$

In other words, the number in each square is equal to the difference of the number of points in the graph of μ and the number of points in the graph of ν that lie above and to the right, i.e., in the North–east (NE) sector, of this square. We will also say that the integer $\mathrm{NE}_{\nu,\mu}(p, q)$ is the *value* of the diagram in the square $[p, p+1] \times [q, q+1]$.

Figure 2.1 represents the diagram of the couple (ν, μ), where $\mu = (5, 6, 4, 2, 1, 3)$ and $\nu = (2, 1, 3, 5, 4, 6)$.

Observe that squares sharing a joint edge have the same value unless they are separated by a line of the planar representation of (ν, μ), in which case the values may differ by ± 1, following the rules shown in Fig. 2.2.

By a rectangle in a diagram, we mean the part of the diagram contained in a subset of the form $[p, p+k] \times [q, q+\ell]$, where $k, \ell \geqslant 1$, $1 \leqslant p < p+k \leqslant n$, and $1 \leqslant q < q+\ell \leqslant n$. We shall need the following lemma in later sections. Its proof is evident.

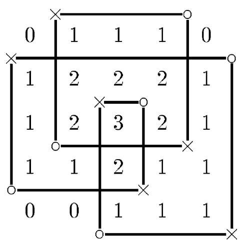


Fig. 2.1.

Fig. 2.2.

2.2. Lemma. *If $\nu \leqslant \mu$, and i, j, i', j' are four values of the diagram of (ν, μ) placed in the four corners of a rectangle as follows*:

$$\begin{array}{ccc} i' & \cdots\cdots & i \\ \vdots & & \vdots \\ j' & \cdots\cdots & j \end{array}$$

Then $j' - i' = j - i +$ number of circles $-$ number of crosses in the rectangle. In particular, the number of crosses in the rectangle is at least $j - i + i' - j'$.

Diagrams have the following important property.

2.3. Proposition. *The values in the diagram of (ν, μ) are all $\geqslant 0$ if and only if $\nu \leqslant \mu$, for the Bruhat order.*

Proof. For all p and $q \in \{1, \dots, n-1\}$ and all permutations $\sigma \in S_n$ we have

$$\mathrm{card}\big\{k \leqslant p \mid \sigma(k) > q\big\} + \mathrm{card}\big\{k \leqslant p \mid \sigma(k) \leqslant q\big\} = p.$$

Therefore, $\mathrm{NE}_{\nu,\mu}(p, q) \geqslant 0$ for all p and q if and only if

$$\mathrm{card}\big\{k \leqslant p \mid \mu(k) \leqslant q\big\} \leqslant \mathrm{card}\big\{k \leqslant p \mid \nu(k) \leqslant q\big\}$$

for all p and q, which is well known to be equivalent to $\nu \leqslant \mu$ (see [Fu], §10.5). □

2.4. Remark. Similarly to NE-diagrams, we may define the NW-diagram, the SE-diagram, and the SW-diagram of a couple (ν, μ) of permutations in S_n by replacing the values $\mathrm{NE}_{\nu,\mu}(p, q)$ respectively by the integers

$$\begin{aligned}
\mathrm{NE}_{\nu,\mu}(p, q) &= \mathrm{card}\big\{k \leqslant p \mid \mu(k) \leqslant q\big\} - \mathrm{card}\big\{k \leqslant p \mid \nu(k) \leqslant q\big\}, \\
\mathrm{SW}_{\nu,\mu}(p, q) &= \mathrm{card}\big\{k > p \mid \mu(k) \leqslant q\big\} - \mathrm{card}\big\{k > p \mid \nu(k) \leqslant q\big\}, \\
\mathrm{SE}_{\nu,\mu}(p, q) &= \mathrm{card}\big\{k > p \mid \mu(k) > q\big\} - \mathrm{card}\big\{k > p \mid \nu(k) > q\big\}.
\end{aligned}$$

Note that $\mathrm{NE}_{\nu,\mu}(p, q) = -\mathrm{NW}_{\nu,\mu}(p, q) = \mathrm{SW}_{\nu,\mu}(p, q) = -\mathrm{SE}_{\nu,\mu}(p, q)$ for all p and q.

3. Proof of Lakshmibai and Sandhya's conjecture

As an application of Theorem 1.3, we prove a conjecture stated by Lakshmibai and Sandhya in [LSa, Section 3].

3.1. The conjecture. Given a permutation $\mu \in S_n$, we define the set Z_μ of permutations $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in S_n$ such that either Condition (1) or Condition (2) below holds:

(1) There exist integers $1 \leqslant i < j < k < \ell \leqslant n$ and $1 \leqslant i' < j' < k' < \ell' \leqslant n$ such that
   (a) $\mu_k < \mu_\ell < \mu_i < \mu_j$ and $\nu_{i'} = \mu_k$, $\nu_{j'} = \mu_i$, $\nu_{k'} = \mu_\ell$, $\nu_{\ell'} = \mu_j$;
   (b) $\nu' \leqslant \nu \leqslant \mu' \leqslant \mu$ for the Bruhat order on S_n, where ν' is the permutation obtained from μ by replacing $\mu_i, \mu_j, \mu_k, \mu_\ell$ respectively by $\mu_k, \mu_i, \mu_\ell, \mu_j$, and μ' is the permutation obtained from ν by replacing $\nu_{i'}$, $\nu_{j'}$, $\nu_{k'}$, $\nu_{\ell'}$ respectively by $\nu_{j'}, \nu_{\ell'}, \nu_{i'}, \nu_{k'}$.
(2) There exist integers $1 \leqslant i < j < k < \ell \leqslant n$ and $1 \leqslant i' < j' < k' < \ell' \leqslant n$ such that
   (a) $\mu_\ell < \mu_j < \mu_k < \mu_i$ and $\nu_{i'} = \mu_j$, $\nu_{j'} = \mu_\ell$, $\nu_{k'} = \mu_i$, $\nu_{\ell'} = \mu_k$;
   (b) $\nu' \leqslant \nu \leqslant \mu' \leqslant \mu$ for the Bruhat order on S_n, where ν' is the permutation obtained from μ by replacing $\mu_i, \mu_j, \mu_k, \mu_\ell$ respectively by $\mu_j, \mu_\ell, \mu_i, \mu_k$, and μ' is the permutation obtained from ν by replacing $\nu_{i'}, \nu_{j'}, \nu_{k'}, \nu_{\ell'}$ respectively by $\nu_{k'}, \nu_{i'}, \nu_{\ell'}, \nu_{j'}$.

Lakshmibai and Sandhya conjectured that the singular locus of X_μ is the union of the subvarieties X_ν, where ν runs over the maximal elements of Z_μ under the Bruhat order. We confirm their conjecture.

3.2. Theorem. *The set* $\mathrm{Msg}(\mu)$ *of maximal singular permutations with respect to* μ *coincides with the set of maximal elements of* Z_μ.

Proof. In [Ga, Theorem 1.4], Gasharov proved that any element of the set Z_μ is singular with respect to μ. It therefore remains to check that any permutation $\nu \leqslant \mu$ that is maximal singular with respect to μ belongs to Z_μ.

By Theorem 1.3 we know that the maximal singular permutations ν with respect to μ fall in three cases. It suffices to check that in each case ν belongs to Z_μ. We shall treat only one case, namely when the planar representation of (ν, μ) is of type $\mathrm{I}(a, b)$. The two other cases can be treated in a similar fashion.

Consider the points $A, \ldots, F$ of Fig. 1.1 and their x-coordinates $x_A, \ldots, x_F$. Set $i = x_B$, $i' = x_A$, $j = j' = x_F$, $k = k' = x_E$, $\ell' = x_D$, and $\ell = x_C$. By definition of type $\mathrm{I}(a, b)$,

$$i \leqslant i' < j = j' < k = k' < \ell' \leqslant \ell.$$

Then $\mu_k < \mu_\ell < \mu_i < \mu_j$ and $\nu_{i'} = \mu_k$, $\nu_{j'} = \mu_i$, $\nu_{k'} = \mu_\ell$, $\nu_{\ell'} = \mu_j$. Let ν' and μ' be the permutations obtained from μ and ν as stipulated in Condition (1)(b) above. Then $\nu < \mu'$ since $1324 < 3412$.

In order to prove that $\mu' \leqslant \mu$, we consider the diagram of (μ', μ). Its planar representation is shown in Fig. 3.1, where the points of the graph of μ' (respectively of μ) are marked by $\times$ (respectively by $\circ$). If a and $b \geqslant 2$, the planar representation has two non-intersecting circuits, each one being a simple closed polygonal arc, and the bounded surfaces inside the circuits are disjoint; there is exactly one circuit, a simple closed polygonal arc, if exactly one of the integers a, b equals 1 and there is no circuit if $a = b = 1$. From the rules given in Section 2, it is clear that the values of the diagram are 0 if they are outside the bounded surfaces inside the circuits and 1 otherwise. The values being all $\geqslant 0$, Proposition 2.3 implies $\mu' \leqslant \mu$.

The inequality $\nu' < \nu$ is proved in a similar way. $\square$

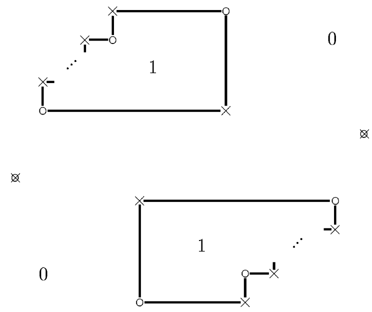


Fig. 3.1. Planar representation of (μ', μ).

4. Proof of Theorem 1.3. Part I

The aim of this section is to prove the "if" part of Theorem 1.3, namely, that (ν, μ) is maximal singular if it has a planar representation of type $\mathrm{I}(a, b)$, $\mathrm{I}(n)$, or $\mathrm{II}(a, b)$.

4.1. Versions

We adopt the following terminology. A *version* of a permutation $\nu \in S_n$ is a pair (A, B) of comparable points in the graph of ν: in other words, if $A = (i, \nu(i))$ and $B = (j, \nu(j))$, then $i < j$ and $\nu(i) < \nu(j)$. The number of inversions of ν being equal to the length $\lg(\nu)$ of ν, it follows that the number of versions of ν is given by $n(n-1)/2 - \lg(\nu)$.

Given a version (A, B) of ν, we may consider the permutation $\nu' = \nu\tau_{i,j}$ obtained as the composition of ν and the transposition $\tau_{i,j}$ exchanging the x-coordinates of A and B. We say that ν' is obtained from ν by flipping the version (A, B) and we write $\nu \to \nu'$. Let us compare the diagrams of ν and ν' with respect to a third permutation μ.

4.2. Lemma. *If $\nu' = \nu\tau_{i,j}$, we have*

$$\mathrm{NE}_{\nu,\mu}(p, q) = \begin{cases} \mathrm{NE}_{\nu',\mu}(p, q) + 1 & \text{if } i \leqslant p < j \text{ and } \nu(i) \leqslant q < \nu(j), \\ \mathrm{NE}_{\nu',\mu}(p, q) & \text{otherwise.} \end{cases}$$

Proof. It follows from the identity $\mathrm{NE}_{\nu,\mu} = \mathrm{NE}_{\nu,\nu'} + \mathrm{NE}_{\nu',\mu}$ and the trivial computation of $\mathrm{NE}_{\nu,\nu'}$. □

4.3. Flippable versions

Consider a couple (ν, μ) of permutations such that $\nu < \mu$ for the Bruhat order. We say that a version (A, B) of ν is *μ-flippable* (or *flippable with respect to* μ, or simply *flippable* if there is no ambiguity about μ) if the permutation ν' obtained from ν by flipping (A, B) satisfies $\nu' \leqslant \mu$, for the Bruhat order.

Flippable versions can be characterized with the help of the diagrams introduced in Section 2.1. We need the following terminology: if (A, B) is a version of ν, we call *rectangle of the version* the part of the diagram of (ν, μ) in the rectangle $[i, j] \times [\nu(i), \nu(j)]$, where $A = (i, \nu(i))$ and $B = (j, (\nu(j))$.

4.4. Lemma. *Let $\nu \leqslant \mu$. A version of ν is μ-flippable if and only if its rectangle has only values $\geqslant 1$.*

Proof. It is a consequence of Lemma 4.2 and Proposition 2.3. □

The following is a useful characterization of singular maximal couples of permutations.

4.5. Proposition. *Given permutations $\nu < \mu$, the couple (ν, μ) is singular maximal if and only if*

(i) *the number of non-flippable versions of ν is less than the number of versions of μ and*
(ii) *for any $\nu' \leqslant \mu$ such that $\nu \to \nu'$, the number of non-flippable versions of ν' is equal to the number of versions of μ.*

Proof. It is clear that (ν, μ) is maximal singular if and only if (ν, μ) is singular and (ν', μ) is not singular for any $\nu' \leqslant \mu$ such that $\nu \to \nu'$. Now by Lakshmibai and Seshadri's Theorem 1 in [LSe] or by Ryan's Theorem II in [Ry], the dimension of the tangent space $T_\nu X_\mu$ of the Schubert variety X_μ at a point in the cell C_ν is equal to the number of transpositions τ such that $\nu\tau \leqslant \mu$. In other words, dim $T_\nu X_\mu$ is equal to the sum of the number of inversions of ν and of the number of flippable versions of ν. Therefore

$$\dim T_\nu X_\mu = n(n-1)/2 - \gamma_\nu,$$

where γ_ν is the number of non-flippable versions of ν. By definition of the singular locus, the couple (ν, μ) is singular if and only if

$$\dim T_\nu X_\mu > \dim X_\mu = \lg(\mu),$$

and it is not singular if and only if $\dim T_\nu X_\mu = \dim X_\mu = \lg(\mu)$. We conclude that (ν, μ) is singular if and only if $\gamma_\nu < n(n-1)/2 - \lg(\mu)$, equivalently if and only if the number of non-flippable versions of ν is less than the number of versions of μ; the couple (ν, μ) is not singular if and only if the numbers of non-flippable versions of ν and of versions of μ are equal. This completes the proof. □

4.6. Corollary. *If the couple (ν, μ) of permutations in S_n is maximal singular, then so are the couples (ν^{-1}, μ^{-1}), $(w_0 \nu w_0, w_0 \mu w_0)$, $(w_0 \nu^{-1} w_0, w_0 \mu^{-1} w_0)$, where w_0 is the longest element $(n, n-1, n-2, \ldots, 2, 1)$ in S_n.*

Our problem is invariant under symmetries; we shall repeatedly use this fact in the sequel.

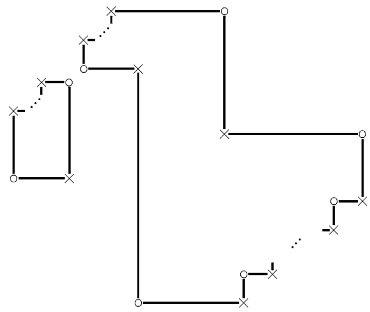


Fig. 4.1. Planar representation of (ν', μ).

4.7. *Proof of the "if" part of Theorem 1.3*

We have to check that, if $\nu < \mu$ has a planar representation of type $\mathrm{I}(a, b)$, $\mathrm{I}(n)$, or $\mathrm{II}(a, b)$, then (ν, μ) is maximal singular. We shall treat the case $\mathrm{I}(a, b)$. The other cases have similar proofs.

Suppose that the couple (ν, μ) has a planar representation of type $\mathrm{I}(a, b)$. By Proposition 4.5 we have to show that the number of non-flippable versions of ν is less than the number of versions of μ and that, after flipping any flippable version of ν, the new permutation has a number of non-flippable versions equal to the number of versions of μ.

The planar representation may have points that are not shown in Fig. 1.1. These are double points sitting outside the circuit of the planar representation. As observed in Section 2, the diagram of (ν, μ) has values equal to 0 outside the circuit. Therefore by Section 2 there are as many crosses as circles in the NW and the SW sectors of any double point. It follows that we need not count the versions not entirely formed with points shown in Fig. 1.1.

A quick count shows that the number of versions of μ, is $ab + a + b - 1$. Using Lemma 4.4, we see that the number of non-flippable versions of ν is ab, which is less than the number of versions of ν since $a, b \geqslant 1$.

Let us flip one of the $2(a + b)$ flippable versions of ν, for instance a version consisting of a point between A and B together with E (notations are as in Fig. 1.1). We obtain a new couple (ν', μ) of permutations whose planar representations is such as in Fig. 4.1 (here $\times$ marks the graph of ν'). This representation has two disjoint circuits (each one is a simple closed polygonal arc). The leftmost circuit has i crosses in its upper left corner. The other one has $a - i - 1$ crosses in its upper left corner. The number of non-flippable versions of ν' is given by

$$i(b + 1) + b + (a - i - 1)(b + 1) + b = ab + a + b - 1.$$

5. Jump permutations

We fix a permutation $\mu \in S_n$ and consider permutations ν smaller than μ for the Bruhat order. By Proposition 4.5 a necessary condition for ν to be maximal singular with respect to μ is that the number of non-flippable versions of ν' is greater than the number of non-flippable versions of ν for some $\nu' \leqslant \mu$ such that $\nu \to \nu'$. If the latter holds, we say that ν is a *jump permutation*. A jump permutation is necessarily singular with respect to μ.

5.1. An injective map

Let $\nu < \nu' \leqslant \mu \in S_n$, where ν' has been obtained from ν by flipping a μ-flippable version (A, B) of ν. Let V (respectively V') be the set of non-flippable versions of ν (respectively of ν') with respect to μ. In this section we wish to investigate when V' has more elements than V, in which case ν is a jump permutation. To this end we construct an injection $V \to V'$ and determine when it is non-bijective.

Let (A', B') be the inversion of ν' obtained by flipping the version (A, B). By convention (see Fig. 5.1) A' (respectively B') lies in the same row as A (respectively as B). Consider the subsets V_1 and V_2 of V defined by

$$V_1 = \bigl\{v \in V \bigm| \operatorname{card}\bigl(v \cup \{A, B\}\bigr) = 4\bigr\} \quad \text{and} \quad V_2 = \bigl\{v \in V \bigm| \operatorname{card}\bigl(v \cup \{A, B\}\bigr) = 3\bigr\}.$$

Here we identified an inversion with its underlying set of points. Since the version (A, B) is flippable, it does not belong to V and we have $V = V_1 \cup V_2$. Similarly, we define subsets of V' by

$$V_1' = \bigl\{v \in V' \bigm| \operatorname{card}\bigl(v \cup \{A', B'\}\bigr) = 4\bigr\} \quad \text{and} \quad V_2' = \bigl\{v \in V \bigm| \operatorname{card}\bigl(v \cup \{A', B'\}\bigr) = 3\bigr\}.$$

We have $V' = V_1' \cup V_2'$.

Let $v \in V_1$. By Proposition 2.3 and Lemma 4.4, its rectangle contains 0 as a value of the diagram of (ν, μ); the pair v is also a version of ν'. By the same reasons, its rectangle contains 0 as a value of the diagram of (ν', μ). Therefore, v belongs to V_1'. This defines an injection $i_1 : V_1 \to V_1'$.

We now construct a map $i_2 : V_2 \to V_2'$. If v belongs to V_2, then it contains a unique point H different from A and from B. We partition V_2 according to the location of H in the six regions (N), (NW), (W), (S), (SE), (E) determined by the rectangle of (A, B) (see Fig. 5.1). Note that H cannot lie elsewhere since it must form a version with either A or B.

If H is in Region (N), then H necessarily forms a version with B. Since it is non-flippable, the rectangle of (H, B) contains 0 as a value. This value lies outside the rectangle

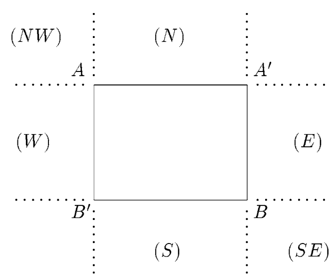


Fig. 5.1.

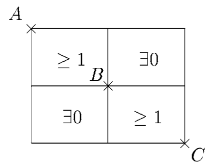


Fig. 5.2.

of (A, B). Therefore after flipping, the version (H, A') is a non-flippable version of ν'. We set $i_2((H, B)) = (H, A')$.

Now let H be in Region (NW). If the rectangle of the version (H, A) contains 0 as a value, then (H, A) and (H, B) are non-flippable versions of ν, and (H, A') and (H, B') are non-flippable versions of ν'. We set $i_2((H, A)) = (H, A')$ and $i_2((H, B)) = (H, B')$. If the rectangle of (H, A) does not contain 0 as a value, then by Lemma 4.4 the version (H, A) is flippable. Therefore, (H, B) is non-flippable. Its rectangle must contain 0 as a value. We map (H, B) to the smallest of the versions (H, A') or (H, B') permitted by the location of the 0-values in the rectangle of (H, B) (smallest means with respect to the natural partial order on $\{1, \ldots, n\}^2$).

If H lies in the remaining four regions, we reduce to the two previous cases by applying the symmetries with respect to the diagonal (A, B) or to the center of the rectangle, which yields a map $i_2 : V_2 \to V_2'$.

5.2. Proposition. (a) *The injective map $i_1 : V_1 \to V_1'$ is not surjective if and only if ν has a flippable version v disjoint from (A, B) such that the intersection of the rectangles of v and of (A, B) contain* 1 *as a value.*

(b) *The map $i_2 : V_2 \to V_2'$ is injective*; *it is not surjective if and only if part of the diagram of (ν, μ) is as in Fig.* 5.2, *the version (A, B) being one of the two flippable versions of ν shown in the figure.*

In Fig. 5.2 three points of the graph of ν are shown. Each rectangle in the figure is a rectangle of the diagram of (ν, μ) as defined in Section 2. The marking $\geqslant 1$ in a rectangle means that the values of the diagram in this rectangle are all $\geqslant 1$. The marking $\exists 0$ in a rectangle means that there is at least one value equal to 0 in this rectangle.

Proof. We first deal with the injective map $i_1 : V_1 \to V_1'$. A version $v \in V_1'$ is also a version of ν. If v is not in the image of i_1, then necessarily the diagram of (ν', μ) contains a value 0 lying both in the rectangle of v and in the rectangle of (A, B). By Lemma 4.2, the diagram of (ν, μ) contains a value 1 lying both in the rectangle of v and in the rectangle of (A, B). The converse is immediate.

The case of $i_2 : V_2 \to V_2'$ must be divided in subcases corresponding to the six regions marked in Fig. 5.1. Again we consider only Regions (N) and (NW). In the case of Region (N) we clearly have a bijection. In the case of Region (NW) the map i_2 is injective since we may recover H. We also have a bijection when the rectangle of the version (H, A) contains 0 as a value. Suppose it does not; in this case, if i_2 is not surjective, then both (H, A') and (H, B') are non-flippable, which means that we are in the situation of Fig. 5.2. $\square$

This leads immediately to the following characterization of jump permutations.

5.3. Corollary. *The permutation ν is a jump permutation if and only if one of Conditions* (i) *or* (ii) *below holds.*

(i) *ν has two disjoint flippable versions whose rectangles intersect and have at least one common value equal to* 1.
(ii) *Part of the diagram of (ν, μ) is as in Fig.* 5.2.

6. The 2143- and 1324-configurations

We fix a permutation $\mu \in S_n$ and consider permutations $\nu < \mu$. In each picture of Fig. 6.1 we represented four points of the graph of ν. As in Fig. 5.2, each rectangle is a rectangle of the diagram of (ν, μ). The marking $\exists 1$ in a rectangle means that the diagram of (ν, μ) has at least one value equal to 1 in this rectangle.

We say that a permutation ν whose diagram has a rectangular part as in the left picture (respectively in the right picture) of Fig. 6.1 has a 2143-*configuration* (respectively a 1324-*configuration*). The union of the rectangles marked by $\geqslant 1$ will be called the *surface* of the configuration.

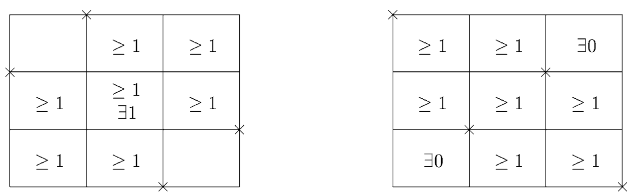
It follows from Corollary 5.3 that any permutation with a 1324-configuration or a 2143-configuration is a jump permutation, hence is singular. This remark will be used frequently in the sequel.

The aim of this section is to prove the following important restriction on maximal singular permutations.

6.1. Theorem. *If (ν, μ) is maximal singular, then ν has a* 1324-*configuration or a* 2143-*configuration.*

In view of Corollary 5.3, Theorem 6.1 is a consequence of the following two lemmas.

6.2. Lemma. *If (ν, μ) is maximal singular and ν has two disjoint μ-flippable versions whose rectangles intersect and have at least one common value equal to* 1*, then ν has a* 2143-*configuration.*



The 2143-configuration The 1324-configuration

Fig. 6.1.

6.3. Lemma. *If* (ν, μ) *is maximal singular and part of its diagram is as in Fig.* 5.2, *then* ν *has a* 1324-*configuration.*

6.4. Proof of Lemma 6.2

Consider the two disjoint μ-flippable versions whose rectangles have a non-empty intersection. Up to symmetry, their rectangles can intersect in the five ways shown in Fig. 6.2.

Since all values in the rectangles are $\geqslant 1$ and the intersection contains at least one value equal to 1, the diagram of (ν, μ) fits into the left picture of Fig. 6.1 or one of the four pictures of Fig. 6.3.

We complete the proof by showing that the pictures of Fig. 6.3 cannot occur under the hypotheses of Lemma 6.2.

Case (a). Consider the version of ν formed by the two crosses in the upper left quarter of the picture corresponding to this case in Fig. 6.3; similarly, consider the version formed by the two crosses in the lower right quarter. Both version are μ-flippable by Lemma 4.4. After flipping both versions, we obtain a permutation ν'' such that $\nu < \nu'' \leqslant \mu$. Its diagram has a part as in Case (d), which proves by Corollary 5.3(i) that ν'' is a jump permutation. Consequently, ν cannot be maximal singular with respect to μ.

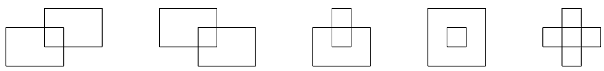


Fig. 6.2.

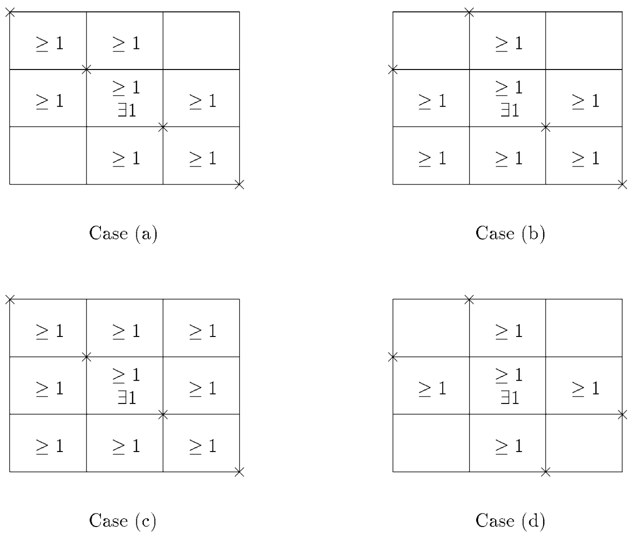


Fig. 6.3.

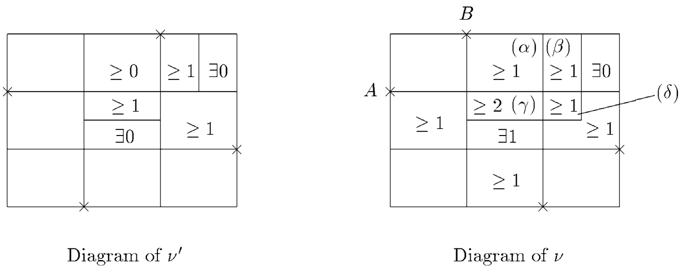


Fig. 6.4.

Case (b). Consider the version of ν formed by the two crosses in the lower right quarter of the picture corresponding to this case in Fig. 6.3. It is flippable by Lemma 4.4. After flipping, we obtain a permutation ν' which is as in Case (d), hence a jump permutation. Therefore, ν cannot be maximal singular with respect to μ.

Case (c). Flip the flippable version formed by the two crosses in the upper left quarter. After flipping, we obtain a jump permutation as in Case (b). We conclude as above.

Case (d). This is the same case as in the left picture of Fig. 6.1, unless the diagram of (ν, μ) has 0 as a value in the upper right rectangle or in the lower left one. Assume that a 0 value is contained in the upper right rectangle. Flip the flippable version of ν formed by the highest cross and the lowest cross. We obtain a permutation ν' whose diagram is of the form shown in the left part of Fig. 6.4: in this picture we have partitioned the upper right and the central rectangles according to the location of a value 0 (the two small rectangles with values $\geqslant 1$ may be empty). Applying Lemma 2.2 with $i = j' = 0$, $j \geqslant 1$, and $i' \geqslant 0$, we see that there exists a point H of the graph of ν' in one of the four rectangles (α), (β), (γ), (δ) shown in the right part of Fig. 6.4; H is also a point of ν.

If H is in (α) or (β), it forms a flippable version with B. After flipping this version, we obtain a permutation which has still a 2143-configuration, hence is a jump permutation by Corollary 5.3(i); hence ν is not maximal singular.

If H is in (γ) or (δ), we flip it with A, and conclude similarly (a 2143-configuration still exists after flipping because the values in (γ) are all $\geqslant 2$). □

6.5. Proof of Lemma 6.3

The points A and B in Fig. 5.2 form a flippable version of ν. After flipping it, we obtain a permutation ν' (see Fig. 6.5 which is Fig. 5.2 after flipping). Consider a 0 value of the diagram of (ν', μ) in the lower left rectangle in Fig. 6.5 and another one in the upper right rectangle. We denote the upper right corner of the square containing the lower 0 value by I and the lower left corner of the square containing the upper 0 value by J as in Fig. 6.5 (we indicated I and J by • in the figure). Applying Lemma 2.2 with $i = j' = 0$, $j \geqslant 1$, and $i' \geqslant 1$, we see that there exists a point H of the graph of ν' in the rectangle whose diagonal is IJ. The point H also belongs to the graph of ν.

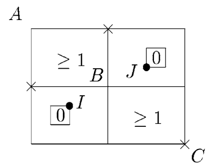


Fig. 6.5.

We claim that H cannot lie in the upper left or lower right rectangles of Fig. 5.2. Indeed, suppose that H lies in the lower right rectangle. Then both (B, H) and (H, C) are flippable versions of ν. One of the permutations (depending on the location of a 0 value in the upper right rectangle) obtained after flipping these versions has a configuration as in Fig. 5.2, hence is a jump permutation. This contradicts the assumption that ν is maximal singular. A similar argument works if H lies in the upper left rectangle.

It follows that H lies either in the rectangle with diagonal IB or in the rectangle with diagonal BJ. Suppose that for each choice of J the point H lies in the latter rectangle. Before we proceed, we introduce some terminology and state a result about partially ordered sets we leave to the reader. An ideal of $\mathcal{P} = \{1, \ldots, k\} \times \{1, \ldots, \ell\}$ (equipped with its usual partial order) is a subset $\mathcal{I}$ such that any element $y \in \mathcal{P}$ satisfying $y \geqslant x$ for some $x \in \mathcal{I}$ belongs to $\mathcal{I}$. An ideal is generated by the set of its minimal elements. Any two elements x and y of $\mathcal{P}$ have a least upper bound, denoted $\sup(x, y)$. The result we have in mind is the following: if $\mathcal{I} \subset \mathcal{H}$ are ideals of $\mathcal{P}$, then either the ideal generated by some $H \in \mathcal{H}$ contains $\mathcal{I}$, or there are two distinct elements $H_1, H_2 \in \mathcal{H}$ (which we may chose to be minimal) such that $\sup(H_1, H_2) \notin \mathcal{I}$.

Now consider the situation where we identify the upper right rectangle in Fig. 6.5 with $\mathcal{P}$ for some k, the origin being at B; the ideal $\mathcal{I}$ is the one generated by all points J as defined above and $\mathcal{H}$ is the ideal generated by all points H; we have $\mathcal{I} \subset \mathcal{H}$. Applying the above result to this situation, we have two cases: In the first one, we find H (necessarily $H \neq B$), and $ABHC$ forms a configuration as in the right picture of Fig. 6.1. In the second case there is a flippable version (H_1, H_2), and after flipping it, the configuration of Fig. 5.2 is still there, so that ν would not be maximal singular.

Suppose now that for some J, the point H lies outside the rectangle spanned by BJ; then for each I, it must lie in the rectangle spanned by IB, and we proceed in a symmetric fashion. $\square$

In the sequel we shall also need the following proposition.

6.6. Proposition. *If (ν, μ) is maximal singular and has a* 1324*-configuration or a* 2143*-configuration, then there is no point of the graph of ν in the surface of the configuration, except possibly in the central rectangle of a* 1324*-configuration.*

Proof. For the 1324-configuration it is a consequence of the claim in the proof of Lemma 6.3.

Suppose ν has a 2143-configuration. Up to symmetry we have only to check that there is no point of ν in the upper middle, the upper right and the central rectangles in Fig. 6.6.

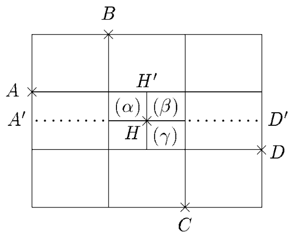


Fig. 6.6.

To deal with the upper middle and upper right rectangles, we argue as in the proof of Lemma 6.2.

Now assume there is a point H of ν in the central rectangle. By hypothesis this rectangle contains a value 1. If this value lies in Rectangle (γ) of Fig. 6.6, then we flip (A, H), which yields a jump configuration $H'A'DC$ satisfying Condition (i) of Corollary 5.3. This contradicts the maximality of ν. By symmetry we may assume that both Rectangles (γ) and (α) contain only values $\geqslant 2$, and that Rectangle (β) contains a value 1. In this case we flip (H, D); we obtain the configuration $BAD'C$ and we conclude similarly with Corollary 5.3(i). □

7. Maximal positive rectangles

Let $\nu \leqslant \mu$. Consider a point $H = (i, \nu(i))$ of the graph of ν such that $\mathrm{NE}_{\nu,\mu}(i, \nu(i)) \geqslant 1$. We call *maximal positive rectangle of* H the unique rectangle of the diagram of (ν, μ) satisfying the following conditions:

(i) its upper left corner is H,
(ii) all its values are $\geqslant 1$,
(iii) the upper row extends as far as possible to the right,
(iv) it extends down as far possible.

By the rules shown in Fig. 2.2, it is clear that the highest segment of length one in the right edge of the maximal positive rectangle is part of a circuit of the planar representation of (ν, μ); by Condition (iii) above, the value immediately to the right of this segment is 0 (unless the rectangle extends to the right edge of $[1, n]^2$). Similarly, at least one segment of length one in the bottom edge is part of a circuit and the value immediately below this segment is 0 (unless the rectangle extends to the bottom edge of $[1, n]^2$). We have summed up this information in Fig. 7.1, where thick lines represent parts of the circuits.

7.1. Proposition. *The maximal positive rectangle of H contains* (*possibly on its boundary*) *another point of ν forming a μ-flippable version with H.*

Proof. Extend the vertical part of the circuit on the right edge of the rectangle as far down as possible. The bottom end of this vertical segment of the circuit must be a point I of the

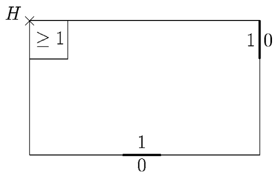


Fig. 7.1.

graph of ν because of the rules edicted in Fig. 2.2. If I is on the right edge of the rectangle, the latter contains two points of the graph of ν, namely H and I.

By definition, we have a square S of the diagram of (ν, μ) which sits on the bottom edge of the rectangle, whose value is 1 and whose lower edge is a horizontal part of a circuit. Extend this horizontal part as far right as possible. It will end in a point J of the graph of ν. If J lies on the bottom edge of the rectangle, we again have two points of the graph of ν in the rectangle.

If I is below the lower right corner K of the rectangle and J is to the right of K, we see that a vertical segment and a horizontal segment of the circuit intersect at K. Applying the rules shown in the first and the third pictures of Fig. 2.2, we see that

$$\mathrm{NE}_{\nu,\mu}(k-1, \ell-1) = \mathrm{NE}_{\nu,\mu}(k, \ell) + 2,$$

if $K = (k, \ell)$. Since $\mathrm{NE}_{\nu,\mu}(k, \ell) \geqslant 0$ by Proposition 2.3, this implies that the value j of the diagram of (ν, μ) immediately to the left and above K satisfies $j \geqslant 2$. By definition the value of the square sitting in the upper row of the rectangle vertically above S is $\geqslant 1$. Applying Lemma 2.2 (with $j \geqslant 2$, $i = j' = 1$, $i' \geqslant 1$) to the rectangle R formed by this square, S and the two squares inside the right corners of the maximal positive rectangle of H, we see that R must contain at least one cross.

In all situations we have a point H' of the graph of ν different from H in the maximal positive rectangle of H. Since there are only positive values in this rectangle, the version (H, H') is flippable. $\square$

The next result is of independent interest. It sheds some light on the Bruhat order.

7.2. Corollary. *Let $\nu < \mu$. A point H of the graph of ν is part of a μ-flippable version of ν if and only if $\mathrm{NE}_{\nu,\mu}(k, \ell) \geqslant 1$ or $\mathrm{NE}_{\nu,\mu}(k-1, \ell-1) \geqslant 1$, where $H = (k, \ell)$. The latter condition always holds if H is a simple point.*

Proof. If $\mathrm{NE}_{\nu,\mu}(k, \ell) \geqslant 1$, we apply Proposition 7.1. If $\mathrm{NE}_{\nu,\mu}(k-1, \ell-1) \geqslant 1$, we apply a symmetrical version of Proposition 7.1. The converse is clear: by Lemma 4.4 any flippable version has a rectangle in which the values of the diagram are positive.

If H is a simple point, then a circuit passes through it. The four configurations of Fig. 7.2 may occur. The values shown follow the rules given in Fig. 2.2. Since they are non-negative by Proposition 2.3, we have the desired conclusion. $\square$

We give a strengthening of Proposition 7.1.

Fig. 7.2.

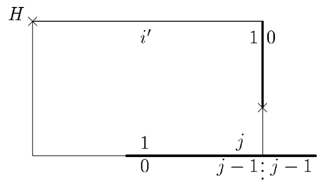


Fig. 7.3.

7.3. Lemma. *Let K be the lower right corner of the maximal positive rectangle of a point H. Assume that $\mathrm{NE}_{\nu,\mu}(k,\ell) \geqslant 1$, where $K = (k,\ell)$. Then the maximal positive rectangle contains* (*possibly on its boundary*) *at least two points* (*different from H*) *of the graph of ν.*

Proof. (a) Suppose that there is no point of ν on the right vertical edge or on the lower horizontal edge of the rectangle. Then we are in the situation dealt with in the proof of Proposition 7.1 and we have $\mathrm{NE}_{\nu,\mu}(k-1,\ell-1) = \mathrm{NE}_{\nu,\mu}(k,\ell)+2$. The hypothesis implies than that $\mathrm{NE}_{\nu,\mu}(k-1,\ell-1) \geqslant 3$. The end of the proof follows the same lines as in the proof of Proposition 7.1.

If there is a point of ν on the right vertical edge of the rectangle and none on the lower horizontal edge, we are in the situation shown in Fig. 7.3. Since $j-1 \geqslant 1$ by hypothesis, we have $j \geqslant 2$ and we may argue as in the proof of Proposition 7.1.

There is a similar proof for the symmetrical case where there is a point of ν on the lower horizontal edge of the rectangle and none on the lower right vertical edge.

If there are points on both edges, we are done. □

8. Forbidden cases

In this section we give sufficient conditions for a singular permutation ν not to be maximal singular with respect to a given permutation μ.

8.1. Lemma. *Suppose ν has a* 1324*-configuration or a* 2143*-configuration and there exists a flippable version (H, H') of ν whose points H and H' are distinct from the four points of the configuration, and whose rectangle does not intersect the surface of the configuration. Then ν is not maximal singular.*

Proof. After flipping (H, H'), we obtain a permutation $\nu' > \nu$, which by Corollary 5.3 is a jump permutation. Therefore ν cannot be maximal singular. □

In the sequel we shall refer to the situations dealt with in Lemma 8.1 as *trivial cases.*

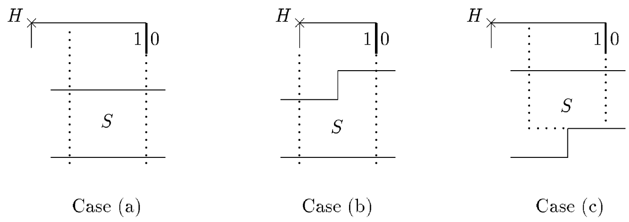


Fig. 8.1.

8.2. Lemma. *Suppose ν has a* 1324*-configuration or a* 2143*-configuration and there exists a flippable version (H, H') of ν such that H and H' are distinct from the four points of the configuration, and the intersection of the rectangle of (H, H') with the surface of the configuration has all values $\geqslant 2$. Then ν is not maximal singular.*

Proof. After we flip (H, H'), the surface will still have values $\geqslant 1$ (Lemma 4.2). We have to concentrate on the $\exists 1$ and $\exists 0$ conditions in Fig. 6.1. By hypothesis a square of the diagram of (ν, μ) with value 1 cannot be in the intersection, and so is not affected by the flipping. By Lemma 4.2 the set of squares with values 0 extends under the flipping. Therefore the permutation obtained from ν by flipping (H, H') is a jump permutation by Corollary 5.3. It follows that ν cannot be maximal singular. □

By Theorem 6.1 we know that a maximal singular permutation ν has a 1324-configuration or a 2143-configuration. In both situations we may speak about the surface of the configuration (defined in Section 6).

There are three pictures in Fig. 8.1. In each there is a point H of ν, the upper part of its maximal positive rectangle, below which one can see a portion of the surface of the configuration. The part of the surface delimited by dots is called S. As before thick lines represent parts of circuits.

8.3. Lemma. *Suppose that ν is maximal singular and that the surface of its configuration has a relative position to some maximal positive rectangle as in one of the pictures of Fig.* 8.1. *In Case* (a) *we assume that the upper edge of the surface shown in the figure is part of the highest horizontal edge of the surface. Then all values in the part S of the surface shown in the figure are $\geqslant 2$.*

Proof. Case (a). By definition the values in the surface S are all $\geqslant 1$. If we exclude the trivial cases (defined above), necessarily the right vertical edge of the maximal positive rectangle crosses S. We claim that this edge crosses S completely. Indeed, if it did not, the rectangle would by Proposition 7.1 contain a point $H' \neq A$. By Proposition 6.6, H' would not belong to the surface of the configuration, hence by our extra hypothesis H' would lie higher then the surface. The flippable version (A, H') would then satisfy the hypotheses of Lemma 8.1, which contradicts the maximality of ν.

Extend the vertical part of the circuit shown by a thick line in Fig. 8.1 as far down as possible. For the same reason as above, it must completely cross S. Consider four values of diagram of (ν, μ) placed at the four corners of a rectangle as in Fig. 8.2. In the proof of

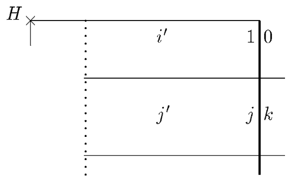


Fig. 8.2.

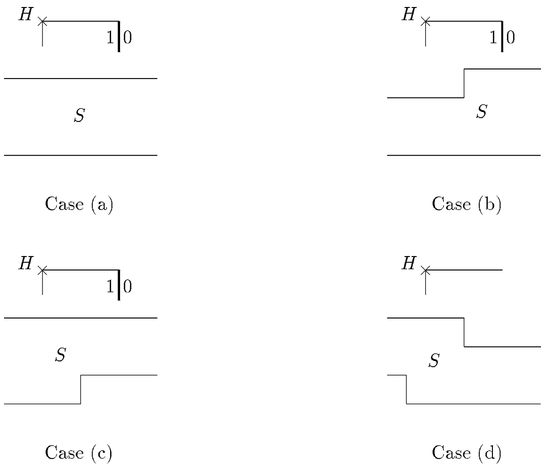


Fig. 8.3.

Proposition 7.1 we saw that the lower end of the vertical part of the circuit is a point of ν. It follows from Fig. 2.2 that $j = k + 1$. Since k is a value of the surface, we have $k \geqslant 1$, hence $j \geqslant 2$. Since i' is a value in the maximal positive rectangle, we have $i' \geqslant 1$. It suffices now to apply Lemma 2.2: $j' > i' \geqslant 1$.

Cases (b) and (c) have analogous proofs. □

8.4. Lemma. *Assume that ν has a* 1324*-configuration or a* 2143*-configuration and that the surface of the configuration has a relative position to some maximal positive rectangle as in one of the four cases of Fig.* 8.3*. Then ν is not maximal singular.*

Proof. Case (a). We apply Lemmas 8.1 and 8.3. By Proposition 7.1 there exists a flippable version (H, H') of ν such that H and H' are distinct from the four points of the configuration, and the intersection of the rectangle of (H, H') with the surface of the configuration has all values $\geqslant 2$. We conclude with Lemma 8.2.

Case (b). The proof is similar to the proof of Case (a).

Case (c). Clearly we have a 2143-configuration. Since the values in the surface S are positive and we exclude the trivial cases, the maximal positive rectangle intersects S as in the left picture of Fig. 8.4. The version (H, C) is flippable. After flipping it, we obtain a new permutation ν' with C replaced by C'. We claim that ν' is a jump permutation,

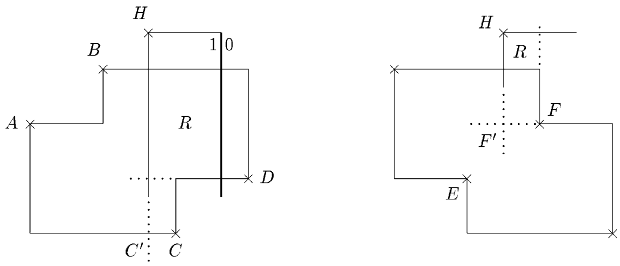


Fig. 8.4.

which implies that ν is not maximal singular. Consider the rectangle R. By Case (c) of Lemma 8.3, all values in R are $\geqslant 2$. Therefore the value 1 existing in the central rectangle of the configuration is not in the rectangle of the version (H, C). Thus it still exists after flipping. Corollary 5.3(i) implies that ν' is a jump permutation.

Case (d). We have a 1324-configuration. Since we exclude trivial cases and Case (a), we are in the situation of the right picture of Fig. 8.4. The values in the rectangle R are all $\geqslant 1$ so that we can flip the version (H, F). After flipping we get a permutation ν' with a 1324-configuration where F is replaced by F'. Therefore ν' is a jump permutation and ν cannot be maximal singular. $\square$

9. Placement of the points of ν

In this section we place the points of the graph of ν when ν is maximal singular with respect to a fixed permutation μ. By Theorem 6.1 we know that necessarily ν has a 1324-configuration or a 2143-configuration.

Figure 9.1 has three boxes filled with crosses ($\times$). The upper left box has $a \geqslant 1$ crosses, the lower right one has $b \geqslant 1$ crosses and the middle one has $n + 2$ crosses with $n \geqslant 0$ (in other words, the middle box has at least two crosses). In each box no two crosses are comparable for the natural partial order on $\{1, \ldots, n\}^2$. On the contrary, any two crosses from different boxes are comparable.

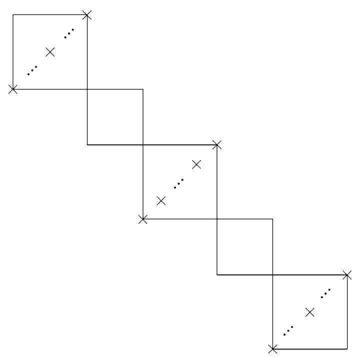


Fig. 9.1.

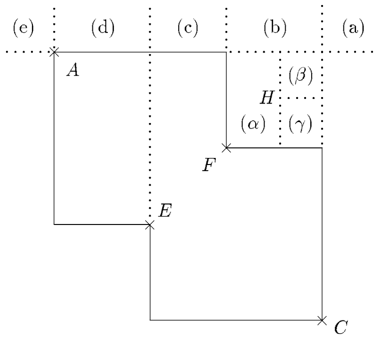


Fig. 9.2.

9.1. Proposition. *If ν is maximal singular and has a* 1324*-configuration, then, except for some double points, the points of ν are as in Fig.* 9.1*. Moreover, if $n \geqslant 1$, then $a = b = 1$.*

Proof. The permutation ν may have several 1324-configurations. We consider one, $AFEC$, represented in Fig. 9.2, such that the central rectangle has the biggest area.

1. To prove the first assertion of the proposition, it suffices up to symmetry to show that there are no simple points of ν in Regions (a)–(c), (e), (f) of Fig. 9.2, that there is no pair of simple points of ν forming a version in Region (d), and that there is no version in the central rectangle of the configuration. Region (f) is the union of Regions (α), (β), (γ) of Fig. 9.2.

Region (a). If there is a simple point in this region, then by Corollary 7.2 it is part of a flippable version, which is excluded by Lemma 8.1.

Region (b). Suppose this region contains a simple point H. If H did not have a maximal positive rectangle, then by a symmetrical version of Proposition 7.1 it would be part of a version (H', H), where H is higher than H. This is impossible in view of Lemma 8.1. Therefore H has a maximal positive rectangle. By Lemma 8.1 and Case (a) of Lemma 8.4, we see that the rectangle extends to the right of C and at least as far down as F. Therefore, (H, C) is flippable. After flipping it, we obtain a permutation ν' satisfying Condition (ii) of Corollary 5.3. Note that the $\exists 0$ condition is preserved under the flipping. Since ν' is a jump permutation, ν cannot be maximal.

Region (c). Argue as for Region (b) using Cases (a) and (d) of Lemma 8.4.

Region (e). If this region contains a simple point H of ν, we consider its maximal positive rectangle (it exists by the argument we used for Region (b)). If the maximal positive rectangle extends to the right at least as far as F, then we may flip (H, A), which yields a new 1324-configuration, with A replaced by A' above A. If it does not extend to the right as far as F and it extends down at least as E, we argue in a similar fashion. Suppose that the maximal positive rectangle does not extend to the right as far as F and downwards as far as E. We apply Lemma 7.3, which in this situation implies the existence of a point H' of ν in the maximal positive rectangle; H' is different from A and is outside the surface of the configuration. This leads to a trivial case, which by Lemma 8.1 contradicts the maximality of ν.

Region (f). Suppose this region contains a point H. Excluding the trivial cases and taking account of Case (a) of Lemma 8.4, we see that the maximal positive rectangle of H extends to the right at least as far as the vertical of C and as far down as to intersect the surface of the configuration. Therefore the values in the rectangle (γ) are all $\geqslant 1$ and the version (H, C) is flippable. If the rectangle (α) has a value equal to 0, we flip (H, C), which yields a jump permutation with a 1324-configuration, namely $AFEC'$, where C' is to the left of C. If the rectangle (α) has all values $\geqslant 1$, then necessarily the rectangle (β) has a 0 value. In this case we have another 1324-configuration, namely $AHEC$ which has a bigger central rectangle than $AFEC$, which contradicts our hypothesis.

Suppose there is a pair (H, H') of simple points of ν forming a version in Region (d). Let H be higher than H'. By Lemma 8.1 and Case (a) of Lemma 8.4 the maximal positive rectangle of H must contain H'. The version (H, H') is flippable and yields an example of a trivial case, which by Lemma 8.1 is impossible.

Suppose there are two points of ν forming a version (H, H') in the central rectangle of the configuration. After flipping it, we have a permutation satisfying Condition (ii) of Corollary 5.3. This is impossible in view of the maximality of ν.

2. We prove the second assertion of Proposition 9.1. Suppose that $n \geqslant 1$ and $a \geqslant 2$. Up to symmetry we may suppose that we are in the situation of Fig. 9.3 where we have drawn the surface of the configuration. The point B is supposed to be the highest simple point of ν. Consider the part of the circuit crossing B: it must look like one of the four pictures of Fig. 7.2. It is clear that the vertical part of the circuit starting from B cannot go upwards; indeed, if it did, it would end with a point of μ which should be connected horizontally to another point of ν, which is impossible in view of the hypothesis on B. Therefore the vertical part at B must point downwards. We claim that the horizontal part of the circuit crossing B cannot point to the left; if it would, then we would have $\mathrm{NE}_{\nu,\mu}(k, \ell) = 0$, where $B = (k, \ell)$ because there are neither simple points of ν, nor simple points of μ in the North–East sector of B; by the rules given in Fig. 2.2 we would have $\mathrm{NE}_{\nu,\mu}(k, \ell - 1) < 0$, which contradicts $\nu \leqslant \mu$ in view of Proposition 2.3. We have proved the claim and shown that the horizontal part of the circuit crossing B points to the right, ending in a point O of μ (represented by $\circ$). The point O must be connected vertically to a simple point of ν (namely a cross of Fig. 9.1). We now discuss according to the position of O.

Starting from the left, the first possibility is shown in Fig. 9.3. Due to the configuration of crosses in Fig. 9.1 and our hypothesis on B, the rectangles R, R', and R'' are not crossed

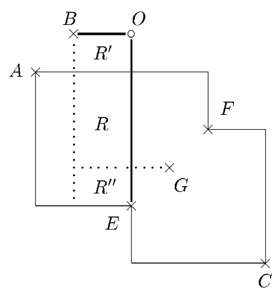


Fig. 9.3.

by any vertical part of a circuit. Therefore the values of the diagram of (ν, μ) stay constant in any row of these rectangles (see Section 2). At their right borders, the values decrease by one when leaving them (see the rules given in Fig. 2.2). The surface of the configuration having positive values, this implies that all values in R and R'' are $\geqslant 2$; all values in the rectangle R' are $\geqslant 1$. This allows us to flip the version (B, E). We then obtain a new 1324-configuration with E replaced by G (the $\exists 0$ condition is preserved). This contradicts the maximality of ν.

If the circle O is vertically above G, then a similar argument as in the previous case shows that we may flip (B, G), yielding a new permutation which keeps the same 1324-configuration $AFEC$. This again is in contradiction with the maximality of ν.

If O is to the right of G, we argue as follows: if the rectangle R in Fig. 9.3 has all values $\geqslant 2$, then we flip (B, E), thus obtaining a 1324-configuration with E replaced by G. If R contains a value 1, we flip (F, C) and obtain the 2143-configuration $BAGE$ (note that R is the central rectangle of this configuration and that the $\exists 1$ condition is satisfied).

The case $n \geqslant 1$ and $b \geqslant 2$ reduces to the previous one by symmetry. □

The next result will be needed in Proposition 9.3.

9.2. Lemma. *If ν is maximal singular and has a* 2143*-configuration and a simple point H as in the left picture of Fig.* 9.4*, then ν has a* 1324*-configuration.*

Figure 9.4 represents the 2143-configuration $BADC$ and its surface.

Proof. By Corollary 7.2 and Lemma 8.1 the point H has a maximal positive rectangle; denote by R (respectively L) its right vertical (respectively lower horizontal) edge. Let us exclude trivial cases. If R is at the left of D, but not of B, and if L is higher than C, but not than A, then we have a 1324-configuration, namely $HBAD$ or $HBAC$ (the right picture in Fig. 9.4 shows such a case).

If R is not at the left of D, we may flip (H, B), which yields a new 2143-configuration. with B replaced by a point above it: this contradicts the maximality of ν. We proceed similarly if L is not above C.

If R is at the left of A, we are led to a trivial case. If R is at the left of B, but not of A, then by what precedes and Lemma 8.1, we may assume that L is as low as A, but not as low as C. Then Lemma 7.3 again leads to a trivial case: indeed there is a point H' of ν, distinct from A, and not in the surface by Proposition 6.6.

It remains to consider the situation where R is at the left of D but not of C; the case when L is above A but not above B is symmetrical to a previous case. So we may assume that L is not higher than D; then Case (c) of Lemma 8.3 implies that the values in the central rectangle are all $\geqslant 2$, a contradiction. □

9.3. Proposition. *If ν is maximal singular and has a* 2143*-configuration, but no* 1324*-configuration, then the simple points of ν are as in Fig.* 9.5.

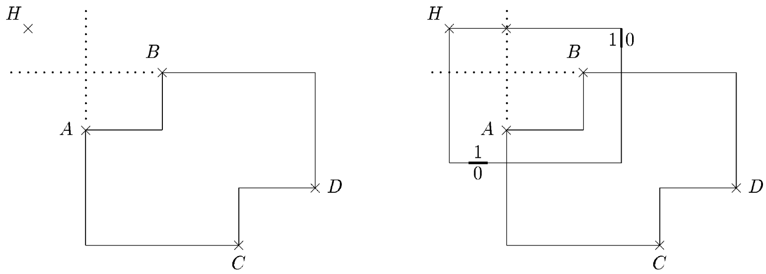


Fig. 9.4.

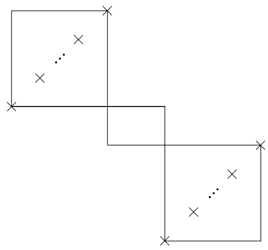


Fig. 9.5.

Figure 9.5 has two boxes filled with crosses ($\times$). Each box has at least two crosses and no two of them are comparable for the natural partial order on $\{1,\dots,n\}^2$. Each cross in the upper left box is smaller than any cross in the lower right box.

Proof. Figure 9.6 shows the 2143-configuration $BADC$ of ν together with regions outside the surface of the configuration. We first prove that there are no simple points of ν in Regions (a), (b), (d), (e) of Fig. 9.6.

Region (a). A simple point H of ν in this region would lead to a trivial case.

Region (b). Suppose we have a simple point H of ν in this region. By Case (a) of Lemma 8.4 and by Lemma 8.1, the maximal positive rectangle of H must extend to the right at least as far as D and must at least touch the surface of the configuration. Flipping the version (H, D), we obtain a new 2143-configuration, which contradicts the maximality of ν.

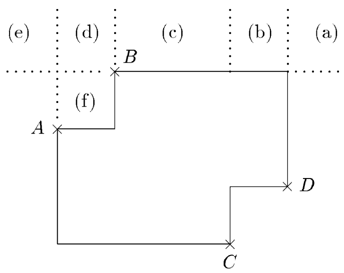


Fig. 9.6.

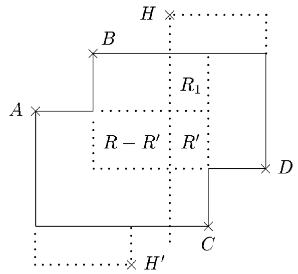


Fig. 9.7.

Region (d). By Case (a) of Lemma 8.4 the maximal positive rectangle of a simple point H in this region must extend to the right at least as far as B. Suppose it does not extend as far as D. If the lower horizontal edge of the rectangle is higher than A, then we apply Lemma 7.3, which ensures the existence of another simple point in the rectangle, different from H and B. Lemma 8.1 shows this is impossible. If the lower horizontal edge of the rectangle is not higher than A, then we may flip (H, B), which yields a new 2143-configuration contradicting the maximality of ν.

If the maximal positive rectangle extends as far as D, it must touch the surface of the configuration by Lemma 8.1. Then the version (H, B) is flippable and after flipping, we have a new 2143-configuration, which again contradicts the maximality of ν.

Region (e) cannot contain a simple point of ν because of Lemma 9.2.

By arguing as in the proof of Proposition 9.1 we show that there are no versions in the interiors of Regions (c) and (f).

To complete the proof of the proposition, it is enough up to symmetry to show that there are no simple points H and H' of ν as in Fig. 9.7 (H and H' are not comparable for the natural order of $\{1, \dots, n\}^2$). Consider the maximal positive rectangle of H. By Cases (a) and (c) of Lemma 8.4, the rectangle must extend to the right at least as far as D and touch the surface of the configuration; hence the rectangle spanned by the diagonal HD has only values $\geqslant 1$. Similarly by symmetry, the rectangle spanned by AH' has only values $\geqslant 1$. If the part R' of the central rectangle R of the configuration has a value 1, then we flip (A, H'), which yields a new 2143-configuration $HA'DC$ with A' above H' and at the right of A. If all values in R' are $\geqslant 2$, there necessarily exists a value 1 in the rectangle $R - R'$. Moreover, all values in the rectangle R_1 are $\geqslant 2$; indeed, if R_1 contains a value $i = 1$, applying Lemma 2.2 with $j \geqslant 2$ in R' and $j' = 1$ in $R - R'$ leads to a contradiction. Therefore we may flip (H, C), which yields a new 2143-configuration $BADC'$ with C' under H and at the left of C. This again contradicts the maximality of ν. $\square$

10. Proof of Theorem 1.3. Part II

Let ν be maximal with respect to μ. Then by Theorem 6.1 it has a 1324-configuration or a 2143-configuration. In this section we show that, if ν has a 1324-configuration, then the planar representation of (ν, μ) is of type $\mathrm{I}(a, b)$ or $\mathrm{I}(n)$. We resume the notation of the beginning of Section 9.

10.1. The case $I(n)$

Let us first prove that, if $a = b = 1$, then the planar representation of (ν, μ) is of type I(n). As in the proof of Proposition 9.1 we consider the 1324-configuration $AFEC$ of ν with the biggest central rectangle. Since $a = b = 1$, there are no simple points of ν outside the surface of the configuration. Consider the horizontal part of the circuit starting from A. It must point to the right and go at least as far as the point A' of Fig. 10.1. Indeed, all values of the diagram of (ν, μ) are $\geqslant 0$ while the values in the surface of the configuration are $\geqslant 1$. Let us prove that the horizontal part of the circuit starting at A ends at A', which therefore is a point of μ. If it ended further to the right, it would have to end at a point of μ which must be connected vertically with a point of ν. So the only other possibility for a point of μ is A''. Let us show this is impossible: Rectangle R in Fig. 10.1 contains a value 0 by definition of a 1324-configuration. Since it cannot be crossed by any circuit, all its values are the same, namely 0. Now all values above $A'A''$ are equal to 0 as well. Therefore the upper edge of R cannot be part of a circuit. Since the horizontal part of the circuit starting at A ends at A', the latter is a point of μ which must be connected vertically with F. By symmetry we see that the boundary of the surface of the configuration is one circuit.

It remains to check that the points of ν that are in the interior of the central rectangle of the configuration are double points. Choose the rightmost simple point B of ν in the interior of the central rectangle. By the above considerations, we know that $\mathrm{NE}_{\nu,\mu}(k, \ell) = 2 - 1 = 1$, where $B = (k, \ell)$. In order to connect B to another simple point, the circuit near B must look as in the second picture of Fig. 7.2. Hence, $\mathrm{NE}_{\nu,\mu}(k, \ell - 1) = 0$, which is impossible for a value in the surface of a configuration. $\square$

Before we deal with the case when $a > 1$ or $b > 1$, we prove three lemmas.

10.2. Lemma. *Suppose ν is as in Proposition* 9.1 *with $n = 0$. Assume that ν has a simple point B higher than A as in Fig.* 10.2. *Then there is no descending vertical part of a circuit whose upper end is B and whose lower end is not higher than C.*

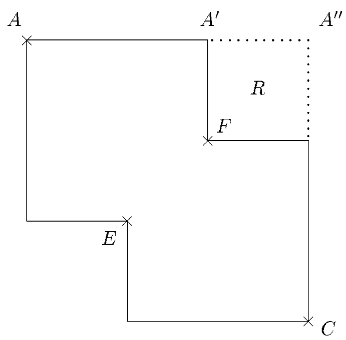


Fig. 10.1.

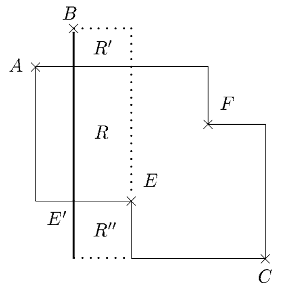


Fig. 10.2.

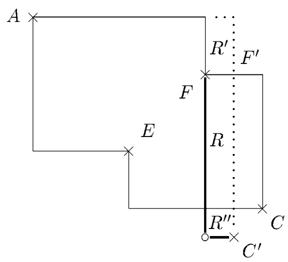


Fig. 10.3.

Proof. Suppose there is a part of a circuit (represented by a thick line) as in Fig. 10.2. Consider the rectangles R, R', R''. They may be crossed by vertical parts of circuits, but only parts whose upper end is a cross. Therefore by Section 2, when sweeping these rectangles from left to right, values may only increase. It follows that all values in R' and R'' are $\geqslant 1$ and all values in R are $\geqslant 2$. Let us flip (B, E). We obtain a new 1324-configuration with E replaced by E'. This contradicts the maximality of ν. $\square$

10.3. Lemma. *Suppose ν is as in Proposition* 9.1 *with $n = 0$. No circuit connects F with the lowest simple point C' of ν as in Fig.* 10.3. *We assume that $C' \neq C$.*

Proof. Suppose there is a part of a circuit as in Fig. 10.3. By the hypothesis on C' the rectangles R, R', R'' in Fig. 10.3 are not crossed by any vertical part of a circuit. It follows that all values in R'' are $\geqslant 1$ and all values in R are $\geqslant 2$. Since R' is not separated from the surface of the configuration by a circuit, the values in R' are $\geqslant 1$ as well. Therefore we may flip (F, C') and we obtain a new 1324-configuration with F replaced by F', which contradicts the maximality of ν. $\square$

10.4. Lemma. *Suppose ν is as in Proposition* 9.1 *with $n = 0$. Assume that C is the lowest simple point of ν. Then no circuit connects F with C as in Fig.* 10.4.

Proof. Consider the value j of the diagram of (ν, μ) in the lower left corner of the surface of the configuration. If we are in the situation shown in Fig. 10.4, there is no simple point of μ in the South–West sector of this corner because such a point could not be connected

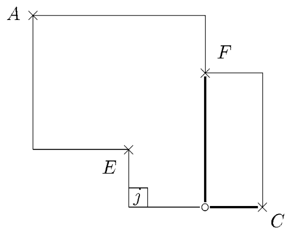


Fig. 10.4.

horizontally to any point of ν. Therefore, by definition of j and by Remark 2.4, we have $j \leqslant 0$, which contradicts the fact that the values in the surface of the configuration are $\geqslant 1$ (Theorem 6.1). $\square$

10.5. The case $I(a,b)$

We now prove that, if $a > 1$ or $b > 1$, then the planar representation of (ν,μ) is of type $\mathrm{I}(a,b)$. Indeed, Proposition 9.1 implies $n = 0$. Consider the lowest simple point C' of ν (we may have $C'' = C$). Arguing as in Part 2 of the proof of Proposition 9.1, we see that the vertical part of the circuit leaving C' goes upwards and the horizontal part of the circuit leaving C' extends to the left (cf. Fig. 10.5). By Lemmas 10.3 and 10.4 the point C' cannot be connected by a circuit with the point F of Fig. 10.4. By Lemma 10.2 it cannot be connected with any simple point of ν higher than A. If it were connected with A or a point of ν to the left of A, then the horizontal part L of the circuit leaving C' would extend to the left of A. By the rules of Fig. 2.2, this would mean that the values of the diagram above L and lower than E would be $\geqslant 1$. This is impossible since a 1324-configuration requires a value 0 in this part of the configuration (see Fig. 6.1). Therefore, the only possibility left is that C' is connected by a circuit with E. By symmetry it follows that we have parts of circuits (drawn in thick lines) as in Fig. 10.5.

Now consider the simple point C'' of ν immediately above C'', but not higher than C. By the same reasons as for C', the horizontal part of the circuit leaving C'' extends to the left. Arguing as for C' implies that C'' can only be connected with C'. By iteration and symmetry this implies that the circuit is as in Fig. 1.1, which means that the planar

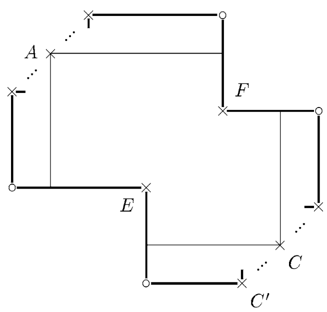


Fig. 10.5.

representation of (ν, μ) is of type $\mathrm{I}(a, b)$. The location of the double points follows from Proposition 6.6.

11. Proof of Theorem 1.3. Part III

In order to complete the proof of Theorem 1.3, we have to show that, if ν is maximal singular with respect to μ and has a 2143-configuration, but no 1324-configuration, then the planar representation of (ν, μ) is of type $\mathrm{II}(a, b)$.

11.1. Lemma. *Suppose ν is maximal singular with respect to μ and does not have any* 1324*-configuration. Assume it has a* 2143*-configuration $BADC$ and a simple point B' as in Fig.* 11.1*. Then there is no vertical part of a circuit entering the surface of the configuration from B'.*

Proof. We argue by contradiction and choose B' to be the highest simple point with the stated property and the 2143-configuration $BADC$ to be the configuration with the highest possible point B. By Lemma 8.1 and Cases (a) and (c) of Lemma 8.4 the maximal positive rectangle at B' exists; it extends to the right at least as far as D and touches the surface of the configuration, so that (B', C) and (B', D) are flippable versions of ν. If there was a value 1 in Rectangle R of Fig. 11.1, then $B'ADC$ would form a 1324-configuration with a higher point than B, which we excluded. Therefore, all values in R are $\geqslant 2$. If all values in Rectangle R' of the figure were $\geqslant 2$, then flipping (B', C) would yield a new 2143-configuration with C replaced by C'', which would contradict the maximality of ν. Therefore R' contains a value 1.

Suppose that the vertical part of a circuit leaving B' enters the surface of the configuration and let O be its lower end (it is a point of μ). If 0 is not higher than A, then by the rules of Fig. 2.2, all values in R' are $\geqslant 2$ unless there is a vertical part L of a circuit entering R' from below and coming from a simple point C' of ν lower than C and to the left of C. We choose the leftmost point C' satisfying this property. This ensures that the values in the parts of Rectangles R, R', R'' left of the vertical line L are all $\geqslant 2$. For similar reasons the values in Rectangle R''' are all $\geqslant 1$ so that (B', C') is a flippable version. Lemma 8.2 shows that this situation is impossible.

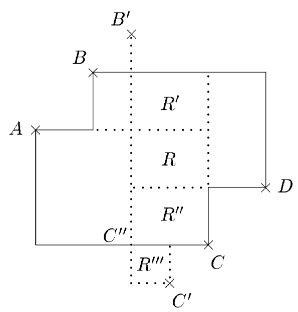


Fig. 11.1.

It follows that the point O of μ must lie higher than A. Since there are no points of ν on the same level as O and on its right, O must be connected via a circuit with a simple point of ν on its left. The existence of a value 1 in R' implies that there is a vertical part L of a circuit entering R' from below and coming from a point C' of ν lower than C and to the left of C. We then argue as above. □

11.2. Lemma. *Suppose ν is maximal singular with respect to μ and does not have any* 1324*-configuration. Assume it has a* 2143*-configuration $BADC$ as in Fig.* 11.1 *and a simple point A' between A and B. Then there is no horizontal part of a circuit entering the surface of the configuration from A'.*

Proof. Suppose there is a horizontal part L of a circuit leaving A' and entering the surface of the configuration. If it does not extend at least as far as C, then it is connected by a circuit either with a point of ν higher than B, or with a point of ν lower than C. Either cases are impossible in view of Lemma 11.1 or a symmetrical version thereof.

Therefore L extends to the right further than C. By the rules of Fig. 2.2 all values of the diagram inside the surface of the configuration and immediately below L must be $\geqslant 2$. Since the central rectangle of the configuration contains a value 1, there exists a horizontal part of a circuit entering the central rectangle from left and coming from a point of ν higher than D and to its right. We are then in a situation symmetrical to one in Lemma 11.1. □

11.3. The case $II(a,b)$

By Proposition 9.3 we know that the simple points of ν are placed as in Fig. 9.5. We consider a 2143-configuration $BADC$ as in Fig. 11.2. Let B' be the simple point of ν immediately above B. By arguing as in Part 2 of the proof of Proposition 9.1, we see that the vertical part of the circuit crossing B' leaves B' downwards. By Lemma 11.1 it cannot enter the surface of the configuration. Therefore B' must be connected to B. The same argument shows that all simple points higher than B are connected with B as in Fig. 11.2.

Let B'' be the simple point of ν immediately below B. For the same reason as for B' the horizontal part of the circuit crossing B'' points to the right. By Lemma 11.2 it cannot enter the surface of the configuration. Therefore B'' must be connected to B. The same argument shows that all simple points lower than B, but not lower than A are connected with B as in Fig. 11.2.

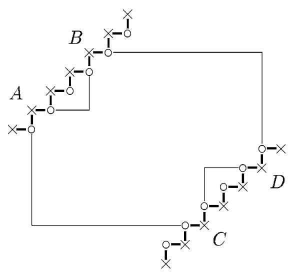


Fig. 11.2.

By symmetry we have parts of circuits as in Fig. 11.2. There are now two ways to complete these into closed circuits. One way yields two disjoint simple circuits, which implies that the values in the surface of the configuration are zero, which is impossible. The other way yields a planar representation of type $\mathrm{II}(a, b)$ as in Fig. 1.3. The location of the double points follows from Proposition 6.6.

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